Set Theory

Def :- A set is a well defined collection of objects (or elements).
By well-defined collection, we mean that given an object we can clearly determine whether the object is the collection or not.

Example :-

① The collection of all vowels of the English alphabet.

{a, e, i, o, u} This is a set.

② The collection of eleven best cricketers in the world.

This is not a set.

* We shall usually denote sets by capital letters, e.g., A, B, C, X, Y, Z. etc.

* We denote the elements by small letters like a, b, c, x, y, z, etc.

Notation :- We write "$a \in A$" (read as "a belongs to A" or "a is an element of A") to say a is an element of the set A.

e.g., Let A be the set of all even natural numbers.

Then $2 \in A$ but $3 \notin A$.

* Representation of Sets :-

Two ways:-

(i) Roster or Tabular form :- all the elements of the set are listed within { } and separated by commas.

e.g.,

① The set A consisting of all even natural numbers less than or equal to 10. will be represented by

$$A = \{2, 4, 6, 8, 10\}.$$

② The set of all odd natural numbers can be represented by

$$\{1, 3, 5, 7, ------\}$$

(ii) Set - builder form :- The set is described by the characteristic property possessed by all the elements of the set.

e.g.,

① The set $A = \{2, 4, 6, 8, 10\}$ is represented in set - builder form as

$A = \{n : n$ is a natural no divisible by 2 and $n \leq 10\}$.

Some notations of Sets used in Mathematics.

$\mathbb{N}$: set of all natural numbers.

$\mathbb{Z}$: set of all integers.

$\mathbb{Q}$: set of all rational numbers

$\mathbb{R}$: set of all real numbers.

$\mathbb{C}$: set of all complex numbers.

$\mathbb{R}^+$: set of all positive real numbers.

$\mathbb{Q}^+$: set of all positive rational numbers.

FREEMIND

Date

Page

The Empty Set: The set containing no elements.

Notation : ϕ , $\{\}$.

The empty set is also called the "null set" or the "void set".

e.g., ① $A = \{n \in \mathbb{N} : 1 < n < 2\}$.

This set is empty because there is no natural number strictly between 1 and 2.

② $B = \{x \in \mathbb{Q} : x^2 = 2\}$

$= \phi$.

* **Finite & Infinite Sets :-**

A set is called a finite set if it contains only finitely many elements; otherwise it is called as infinite set.

e.g. A = set of all vowels is a finite set.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are infinite set.

* **Equal Sets :-**

Two sets A and B are said to be equal if they contain exactly the same elements.

Note that while representing a set the order of the ~~set~~ elements is not important. For Example, if

$A = \{1, 2, 3\}$ & $B = \{2, 3, 1\}$.

Then $A = B$.

* **Subsets :**

Let X be a set. We say that a set A is a subset of X (denoted by $A \subseteq X$).

if every element of A is also ~~present~~ an element of X.

$A \subseteq X$ if $a \in A \Rightarrow a \in X$.

Note that :-

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

* Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

* Power Sets :-
Given a set A, the power set of A is the collection of all subsets of A.

Notation :- $P(A)$ denotes the power set of A, so

$$P(A) = \{ B : B \subseteq A \}.$$

Let $A = \{1, 2, 3\}$

Then $P(A) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$.

Here the number of elements in the

$$P(A) = 8 = 2^3.$$

For a set A, we denote by $|A|$ the no. of elements in A.

If $|A| = n$, then $|P(A)| = 2^n$.

If B is any subset of A, then any particular element of A is either in B or not in B.
There are n elements, and for each we get a subset by specifying whether it is in B or not.

$\Rightarrow |P(A)| = 2^n$.

* Operations on Sets

① Union :- Given two sets A & B, the union of A & B (denoted by $A \cup B$) consists of all the elements which are either in A or in B. (include elements which are in both A & B as well).

$A \cup B = \{x : x \in A$ or $x \in B\}$.

e.g., $A = \{1, 2, 3\}$; $B = \{2, 3, 4, 5\}$.

$A \cup B = \{1, 2, 3, 4, 5\}$.

Note that while representing a set we do not repeat the elements.

② Intersection :- All elements which are common to A & B (denoted by $A \cap B$).

$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

e.g., $A = \{1, 2, 3\}$; $B = \{2, 3, 4, 5\}$

$A \cap B = \{2, 3\}$.

- If $A \cap B = \phi$, then we say that A & B are disjoint.

③. Set difference :-

Notation : $A \setminus B$ or $A - B$.

$A - B = \{x : x \in A \text{ and } x \notin B\}$.

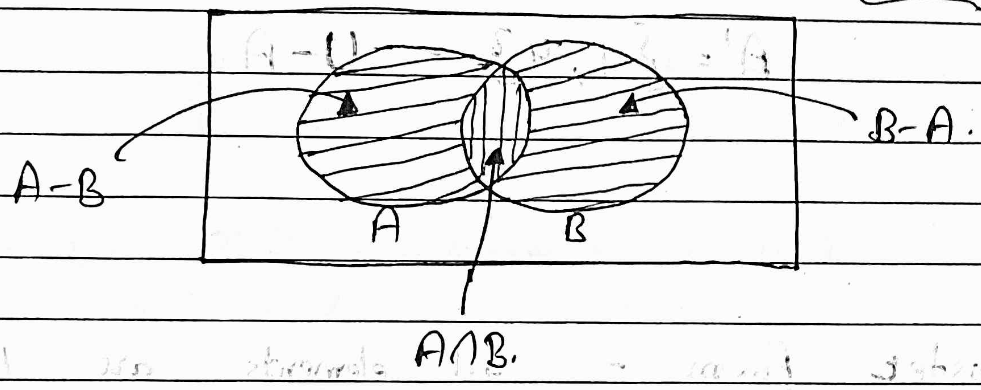
e.g., $A = \{1, 2, 3\}$; $B = \{2, 3, 4, 5\}$.

$A - B = \{1\}$. ; $B - A = \{4, 5\}$.

Note that :

$$A \cup B = (A-B) \cup (A \cap B) \cup (B-A).$$

Disjoint Union



* Singleton Set :

Set containing only one element.

e.g., $\{1\}, \{0\}, \{a\}$.

* Complement of a set :

Let U be an universal set. & A be a subset of U. The complement of a in U is

$$A' = U - A.$$

complement of A consists of all elements which are not in A.

FREEMIND

Date

Page

e.g., $U = \{1, 2, 3, 4\}$

$A = \{2, 3\}$

$A' = \{1, 4\} = U - A$

- Roster Form - all elements are listed.
- set - builder Form -

Note that in general, for infinite set like $\mathbb{R}$ it is not possible to use the roster form.
So, we need set - builder form.

* Notations :-

Intervals ~~of~~ in $\mathbb{R}$. Let $a, b \in \mathbb{R}$ with $a < b$.

$(a, b) = \{x : x \in \mathbb{R}$ & $a < x < b\}$.
→ Open interval.

$[a, b] = \{x : x \in \mathbb{R}, a \leq x \leq b\}$.
→ Closed interval.

$(a, b] = \{x : x \in \mathbb{R}, \quad a < x \leq b\}$.

$[a, b) = \{x : x \in \mathbb{R}, \quad a \leq x < b\}$.

$(a, \infty) = \{x : x \in \mathbb{R}, \quad x > a\}$.

$[a, \infty) = \{x : x \in \mathbb{R}, \quad x \geq a\}$

$(-\infty, b) = \{x : x \in \mathbb{R}, \quad x < b\}$.

$(-\infty, b] = \{x : x \in \mathbb{R}, \quad x \leq b\}$.

$(-\infty, \infty) = \mathbb{R}$

All these intervals are subsets of $\mathbb{R}$.

* **Proper Subset**: We say A is a proper subset of X if A is a subset of X but $A \neq X$.

Notation: $A \subsetneq X$.

This means that ~~A~~ every elements of A is in X, and there is atleast one element in X which is not in A.

e.g., $\{1, 2\} \subsetneq \{1, 2, 3\}$.

* <u>Superset</u> : We say B is a superset of A if A is a subset of B.

Notation : $B \supseteq A$.

$B \supsetneq A$ means $B \supseteq A$ & $B \neq A$.

Properties of Union and Intersection :

* <u>For Union</u> :

(i) $A \cup B = B \cup A$ (Commutative law).

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law).

(iii) $A \cup \phi = A$ (Identity law)

(iv) $A \cup A = A$ (Idempotent law).

(v) If $A \subseteq U$ then $A \cup U = U$

* <u>For Intersection</u> :

(i) $A \cap B = B \cap A$ (commutative law)

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$. (Associative law)

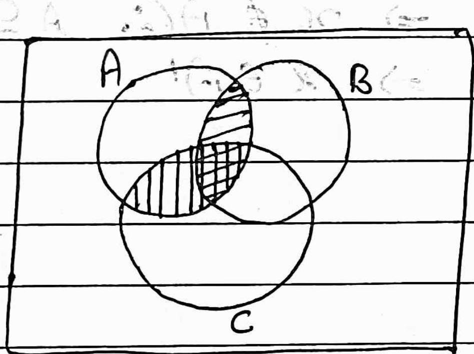
(iii) $A \cap \phi = \phi$. (Identity law)

(iv) $A \cap A = A$. (Idempotent law).

(v) If $A \subseteq B$ then $A \cap B = A$.

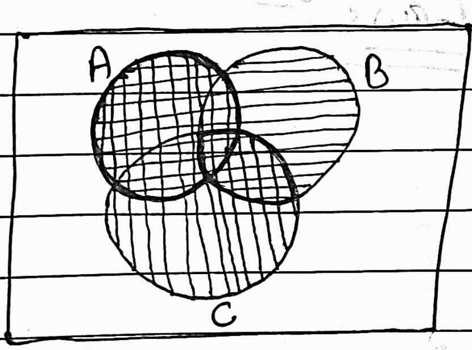
Distributive Law :-

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



$\equiv A \cap B$

$|||\ A \cap C$.



$\equiv A \cup B$

$|||\ A \cup C$

Properties of Complement :-

U : universal set.

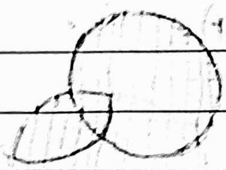
$A' = U - A$

- $(A')' = A$ $\quad [\,(A')' = U - A'$
$= U - (U - A) = A\,]$.

- $\phi' = U$; $U' = \phi$

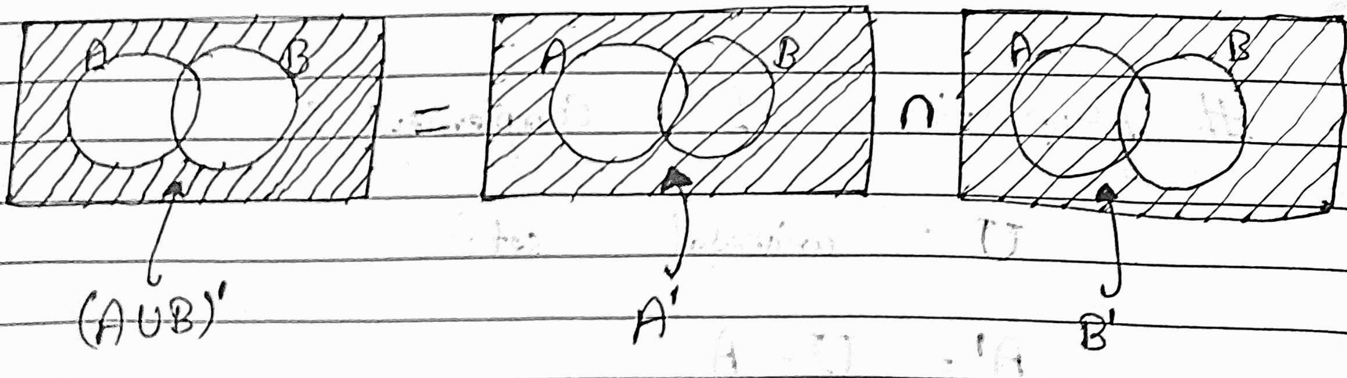
- If $A \subseteq B \subseteq U$, then
$B' \subseteq A'$ $[\,x \in B' \Rightarrow x \notin B$
$\Rightarrow x \notin A$ ($\because A \subseteq B$).
$\Rightarrow x \in A'\,]$

De Morgan's Laws :-



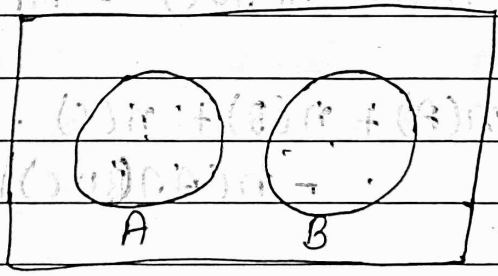
(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$



Let A and B are two finite sets such that $A \cap B = \phi$.

Then $n(A \cup B) = n(A) + n(B)$.



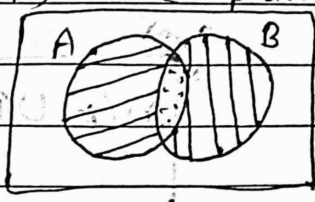
* In general,

Imp:

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

Proof :- $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

where $(A-B)$, $A \cap B$ & $(B-A)$ are pairwise disjoint.



$n(A \cup B) = n(A-B) + n(A \cap B) + n(B-A)$.

$\equiv$ $A-B$

$\Rightarrow n(A \cup B) = [n(A-B) + n(A \cap B)] + [n(B-A) + n(A \cap B)] - n(A \cap B)$

$\therefore$ $A \cap B$

$|||$ $B-A$.

$\Rightarrow \boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$

$\because A = (A-B) \cup (A \cap B) \leftarrow$ disjoint Union

$\therefore n(A) = n(A-B) + n(A \cap B)$.

Illy, $n(B) = n(B-A) + n(A \cap B)$

$n(A \cup B \cup C) = ?$

$$n(A \cup (B \cup C)) = n(A) + n(B \cup C) - n(A \cap (B \cup C))$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap (B \cup C)) \quad \text{--- (1)}$$

By distributive property,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

So, $n(A \cap (B \cup C)) = n(A \cap B) + n(A \cap C) - n((A \cap B) \cap (A \cap C))$

$$= n(A \cap B) + n(A \cap C) - n(A \cap B \cap C) \quad \text{--- (2)}$$

By adding (1) & (2)

imp.

$$\boxed{n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)}$$

Exp. 400 people who speak either English or Hindi or both.

250 people : Speak Hindi

200 people : speak English

FREEMIND

Date

Page

How many speak both language.

Let H : Set of people speaking Hindi

E : Set of people speaking English.

Given :- $n(H) = 250$, $n(E) = 200$.

$n(H \cup E) = 400$.

To find : $n(H \cap E) = ?$

$n(H \cup E) = n(H) + n(E) - n(H \cap E)$.

$n(H \cap E) = n(H) + n(E) - n(H \cup E)$

$n(H \cap E) = 250 + 200 - 400$

$n(H \cap E) = 50$ Answer

Problems :-

① Suppose $A \cup B = A \cup C$ & $A \cap B = A \cap C$.

Prove that $B = C$.

Proof :- To show $B = C$ it is enough to show that $B \subseteq C$ & $C \subseteq B$.

Let $x \in B \subseteq A \cup B = A \cup C$

$\Rightarrow x \in A$ or $x \in C$

If $x \in C$, then O.K.

If $x \in A$ then $x \in A \cap B$ ($\because x \in B$ also).

But $A \cap B = A \cap C$,

so $x \in A \cap C \Rightarrow x \in C$.

So, we have $x \in B \Rightarrow x \in C$

$\therefore B \subseteq C$

Similarly, $C \subseteq B$.

$\therefore B = C$

Another Proof:

$$(A \cup B) = (A \cup C)$$

$$(A \cup B) \cap C = (A \cup C) \cap C$$

$$(A \cap C) \cup (B \cap C) = C$$

$$(A \cap B) \cup (B \cap C) = C \quad \text{--- (1)}$$

$$(A \cup B) = (A \cup C)$$

$$(A \cup B) \cap B = (A \cup C) \cap B$$

$$B = (A \cap B) \cup (B \cap C) \quad \text{--- (2)}$$

from (1) & (2)

$B = C$

2. If $P(A) = P(B)$ then $A = B$.

$P(A) = \{C : C \subseteq A\}$.

Since, $A \subseteq A$, $A \in P(A) = P(B)$

$\Rightarrow A \in P(B) \Rightarrow A \subseteq B$.

Illy, $B \in P(B) = P(A) \Rightarrow B \subseteq A$.

Hence, $A = B$.

Notations :-

Statement 1 $\Rightarrow$ Statement 2.

means if statement 1 is true then statement 2 is true.

Statement 1 $\Leftrightarrow$ Statement 2.

means statement 1 is true if and only if statement 2 is true.

Shorthand : iff $\equiv$ if and only if.

Exp: Suppose $A \cap B \neq \phi$, $B \cap C \neq \phi$,

$$A \cap C \neq \phi.$$

Is it true that $A \cap B \cap C \neq \phi$. ?

Ans:- No.

Let $A = \{0, 1\}$, $B = \{1, 2\}$.

$$C = \{0, 2\}.$$

$0 \in A \cap C$, $1 \in A \cap B$, $2 \in B \cap C$.

But $A \cap B \cap C = \phi$.

Recall:

$n(A) =$ no. of elements in A.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

Also, $n(A) = n(A - B) + n(A \cap B)$.

FREEMIND

Date

Page

Prob 1. $n(A) = 6, \quad n(B) = 4.$

what is the minimum & maximum value of $n(A-B)$?

$$n(A) = n(A-B) + n(A \cap B).$$

$$\Rightarrow n(A-B) = n(A) - n(A \cap B).$$

$$= 6 - n(A \cap B).$$

$\because$ $A \cap B \subseteq B \Rightarrow n(A \cap B) \leq n(B) = 4.$

$n(A-B) \geq 6 - 4 = 2 \leftarrow$ Min. value.

Also, if $A \cap B = \phi$, $n(A \cap B) = 0.$

$\therefore$ Max $n(A-B) = 6 - 0 = 6. \leftarrow$ Max. value.

Prob 2. Suppose there are 60 people

25 read newspaper H.

26 read newspaper I

26 read newspaper T

9 read both H and I.

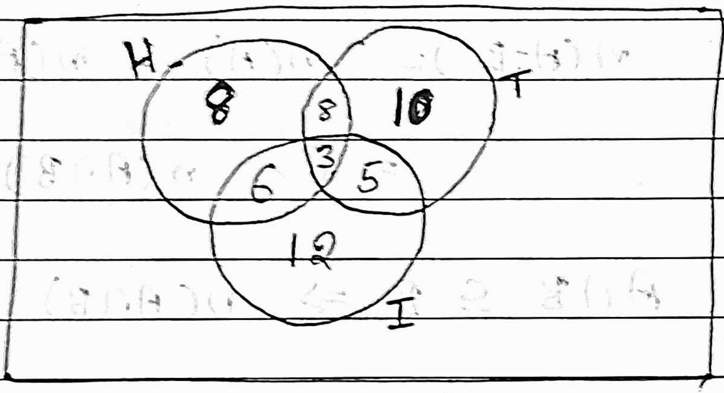
11 read both H and T.

8 read both T and I.

Also, 3 read all three.

Find (i) no. of people who read atleast one of the three.

(ii) no. of people who read exactly one newspaper.



(i) $n(H \cup T \cup I) = 25 + 10 + 5 + 12$

$= 52.$

Or, use $n(H \cup T \cup I)$

$= n(H) + n(T) + n(I) - n(H \cap T) - n(H \cap I) - n(T \cap I) + n(H \cap T \cap I)$

$= 25 + 26 + 26 - 11 - 8 - 8 + 3$

$\Rightarrow 52.$

(ii) From the Venn Diagram.

#. no. of people who reads exactly one = $8 + 10 + 12 = 30.$

Prob 3. Let $n(A \cap B) = x$, $n(A-B) = 6x$.

$n(B-A) = 8+2x$, $n(A) = n(B)$.

find x

Soln. :- $n(A) = n(A-B) + n(A \cap B) = 6x + x = 7x$.

$n(B) = n(B-A) + n(B \cap A) = 8 + 2x + x = 8 + 3x$.

$\therefore$ $7x = 8 + 3x$. $\Rightarrow$ $\boxed{x = 2.}$

Prob 4. Suppose 70% Indians like apple, 82% like mango.

Let x% like both.
find the min. & max. possible x.

Sol^n: Given : $n(A) = 70$, $n(M) = 82$.

$n(A \cap M) = x$.

Since $n(A \cup M) \leq 100$

$n(A) + n(M) - n(A \cap M) \leq 100$

$\Rightarrow$ $70 + 82 - x \leq 100$

$\Rightarrow$ $x \geq 52$.

FREEMIND

Date

Page

Also, $A \cap M \subseteq A$.

$\therefore \quad n(A \cap M) \leq n(A) = 70.$

So, $\boxed{52 \leq x \leq 70.}$

- Recall: $P(A)$ = all subsets of A.

$$n(P(A)) = 2^{n(A)}.$$

* Prob: 5 $\quad n(P(P(P(\phi)))) = ?$

Soln: since $n(\phi) = 0$, $n(P(\phi)) = 2^0 = 1$

$\Rightarrow \quad n(P(P(\phi))) = 2^1 = 2$

$\Rightarrow \quad n(P(P(P(\phi)))) = 2^2 = 4.$

* Prob: 6 $\quad n(A) = m$, $n(B) = n$,

$n(P(A)) - n(P(B)) = 112.$

find m and n.

Soln: $2^m - 2^n = 112 \quad \Rightarrow \quad m > n.$

$$2^n(2^{m-n} - 1) = 112 = 2^4 \times 7.$$

$m = 4$, $2^{m-n} - 1 = 7$.

$2^{m-n} = 8 = 2^3$

$m - n = 3 \Rightarrow m = 3 + n$

$\Rightarrow \boxed{n = 4 , m = 7}$

Prob. 7 $A = \{1, 2, 3, 4\}$; $B = \{2, 4, 6\}$

find the no. of sets C such that

$A \cap B \subseteq C \subseteq A \cup B$.

Sol^n: $A \cap B \{2, 4\}$; $A \cup B = \{1, 2, 3, 4, 6\}$

So, $\{2, 4\} \subseteq C \subseteq \{1, 2, 3, 4, 6\}$.

$\Rightarrow C = \{2, 4\} \cup C'$

where $C' \subseteq \{1, 3, 6\}$.

$\therefore$ # of such C = # of $C' \subseteq \{1, 3, 6\}$.

$= n(P(\{1, 3, 6\}))$

$= 2^3 = 8$.

Prob: (8) Let $X = \{4^n - 3n - 1 : n \in \mathbb{N}\}$

$Y = \{9(n-1) : n \in \mathbb{N}\}$.

Then $X \cup Y$ equals.

(a) $\mathbb{N}$ (b) $Y - X$ (c) X (d) Y.

Sol^n: $X = \{0, 9, 54, \ldots\}$.

$Y = \{0, 9, 18, 27, \ldots\}$.

Clearly, $1 \notin X \cup Y$.

So, (a) is False.

Also, $Y \not\subset X$ ($\because 18 \in Y$, $18 \notin X$).

$\therefore X \cup Y \neq X$. So, (c) is False.

$Y - X \neq X \cup Y$ because $0 \in X \cup Y$, but $0 \notin Y - X$.

By elimination (d) must be true if it is given that one of the statement is true.

Let's try to prove this.

Claim: $X \subseteq Y$ $(\Rightarrow X \cup Y = Y)$

i.e. $4^n - 3n - 1$ is divisible by 9
for all $n \in \mathbb{N}$

Note that for $n = 1$: $4^n - 3n - 1 = 0$.
which is divisible by 9.

Suppose $4^K - 3K - 1$ is divisible by 9.

We will show that $4^{K+1} - 3(K+1) - 1$ is
also divisible by 9.

$4^{K+1} - 3(K+1) - 1$

$\Rightarrow \underbrace{4(4^K - 3K - 1)}_{\text{divisible by } 9} + \underbrace{9K}_{\text{divisible by } 9.}$

So, the sum is divisible by 9.

$\therefore X \subseteq Y.$

Hence, $\boxed{X \cup Y = Y.}$

Prob: 9 $A = \{n^3 + (n+1)^3 + (n+2)^3 : n \in \mathbb{N}\}$,

$B = \{9n : n \in \mathbb{N}\}$.

Then which of the following is/are true.

FREEMIND

Date

Page

(a) $A \subset B$ (b) $B \subseteq A$ (c) $A = B$ (d) $A \subsetneq B$.

Sol^n: $A = \{36, (2^3+3^3+4^3), \ldots\}$

Smallest no. in A is 36.

So, $A \neq B$ and $B \not\subset A$.

(b) & (c) are false.

Claim: $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9 for every $n \in \mathbb{N}$

$$n^3 + (n+1)^3 + (n+2)^3$$

$$\Rightarrow n^3 + (n^3 + 3n^2 + 3n + 1) + (n^3 + 6n^2 + 12n + 8)$$

$$\Rightarrow 3n^3 + 9n^2 + 15n + 9.$$

$$\Rightarrow \underbrace{9(n^2+1)}_{\text{divisible by } 9} + 3n(n^2+5).$$

Another claim: $n(n^2+5)$ is a multiple of 3.

If $n = 3K$, then O.K.

If $n = 3K+1$, $n^2+5 = (3K+1)^2 + 5$

$(3K+1)^2 + 5 = 9K^2 + 6K + 6.$
which is a multiple of 3.

Also, $(3K+2)^2 + 5 = 9K^2 + 12K + 4 + 5$
$= 3(3K^2 + 4K + 3).$
which is a multiple of 3.

Hence, $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9 for all $n \in \mathbb{N}$.

So, $A \subsetneq B$ Hence (a) & (d) are correct

$\Downarrow$

$A \subseteq B$ also.